Int. J. Solids Structures Vol. 26, No. 5/6, pp. 539–547, 1990
Printed in Great Britain.
0020-7683/90 \$3.00 + .00

INFLUENCE OF RIB STIFFENERS ON THE BUCKLING STRENGTH OF ELASTICALLY SUPPORTED TUBES

I. D. MOORE
Geotechnical Research Centre, The University of Western Ontario, London, Canada N6A 5B9

(*Received* 7 *December* 1988)

Abstract—The linear elastic buckling problem is evaluated for elastically supported tubes stiffened with regularly spaced rigid circumferential ribs. The harmonic stiffness equations are described for the shell and the elastic continuum. The linear eigenvalue problem is solved to determine the influence on critical hoop thrust of axial thrust, continuum parameters, interface condition and stiffener spacing.

INTRODUCTION

Flexible cylinders are buried to form pipelines and culverts under road and railway embankments. The design of these buried structures must include an assessment of buckling strength. Various theoretical and experimental studies have been undertaken to determine the elastic stability for plane strain conditions (Forrestal and Herrmann, 1965; Allgood and Ciani, 1968; Crabb and Carder, 1985; Moore and Booker, 1985). Linear buckling theory appears to provide a useful measure of buckling strength for buried structures (Gumbel, 1983; Moore, 1989).

In practice, flexible pipes may respond under three-dimensional conditions. For example, a pipe can be stiffened with circumferential ribs. Buckling solutions are needed for such three-dimensional conditions (Selig and Nash, 1988). A three-dimensional linear buckling solution is described in this paper, which is used to examine how ribs affect buckling strength.

PROBLEM STATEMENT

Figure 1 shows the flexible tube of radius a, thickness t, Young's modulus E and Poisson's ratio ν. For a plane tube, flexural rigidity D and membrane rigidities E_p and G_p are defined by

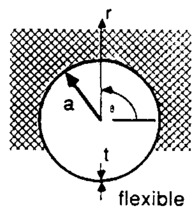
$$D = \frac{Et^3}{12(1-\nu^2)}$$

$$E_p = \frac{Et}{(1-\nu^2)}$$

$$G_p = \frac{Et}{2(1+\nu)}.$$

Alternative expressions exist for corrugated structures, although corrugations will introduce structural anisotropy which complicates the analysis.

The structure is stiffened with circumferential ribs at spacings L. For this study the ribs are assumed to resist all radial movement, but have no torsional stiffness so that they permit non-zero gradients of radial displacement in the axial direction (this might correspond to an open channel stiffener with high flexural and axial stiffness, but low torsional stiffness). Future studies can deal with more complex conditions if the results from the present study indicate that the effect of rib-stiffening is worthy of further attention.



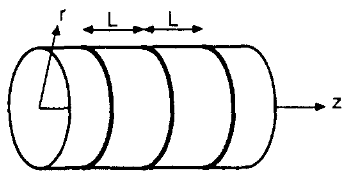


Fig. 1. Elastically supported tube. (a) Cross-sectional view. (b) Rib stiffeners.

The structure is supported at its external surface by an elastic continuum with shear modulus G_s and Poisson's ratio ν_s. Two idealized conditions are considered at the soil–structure interface. Firstly, a bonded interaction can be modelled by enforcing full compatability of ground and structural displacements and equilibrium of tractions. Alternatively, smooth interaction is modelled by setting shear tractions to zero and allowing differential displacements in the circumferential and axial directions.

Hoop thrust N and axial thrust T develop in the structure which can cause elastic instability. Although the presence of rib stiffeners may affect the distribution of stress resultants, for this study the thrusts are assumed to be uniformly distributed. The analysis of Hoeg (1968) or Einstein and Schwartz (1979) for unstiffened elastically supported tubes could be used to determine the stress resultants.

The assumption of zero torsional rigidity is conservative, as is the evaluation of stability for uniform distributions of hoop and axial thrust, as determined using unstiffened elastically supported tube theory (the ribs themselves will carry some of the loads applied to the structure, and thrusts should be reduced). The analysis being presented will be unconservative in that the solution is developed for completely rigid stiffeners which permit no radial displacement around the circumference and for corrugated structures, the bending stiffness D along the structure will be less than that circumferentially (if circumferential bending stiffness for the corrugated profile is used for D, then the buckling strength will be closer to plane strain theory than that predicted using the isotropic analysis).

BUCKLING ANALYSIS

Structural stiffness

Conventional thin shell theory can be used to determine the harmonic stiffness of the structure. The theory of Herrmann and Armenakas (1962) for the three-dimensional harmonic response of a prestressed cylinder is employed for this study. Harmonic radial, circumferential and axial displacements are, respectively,

$$\begin{aligned} w &= W\cos n\theta \cos mz \\ v &= V\sin n\theta \cos mz \\ u &= U\cos n\theta \sin mz \end{aligned} \qquad (1)$$

for circumferential co-ordinate θ and axial co-ordinate z. The traction normal to the shell is

$$r = R\cos n\theta \cos mz \tag{2}$$

and the shear tractions in the circumferential and axial directions are, respectively,

$$\begin{aligned} q &= Q\sin n\theta \cos mz \\ s &= S\cos n\theta \sin mz. \end{aligned} \tag{3}$$

The stiffness equations are then

$$(R, Q, S)^{\mathrm{T}} = K(W, V, U)^{\mathrm{T}}. \tag{4}$$

The matrix K has static A and stability B components

$$K = A + \lambda B \tag{5}$$

for load factor λ.

The static matrix $A = (a_{ij})$ is given by Herrmann and Armenakas (1962):

$$\begin{aligned}
a_{11} &= \frac{-E_p}{a^2} - \frac{D(1-n^2)^2}{a^4} - Dm^4 - \frac{2Dm^2n^2}{a^2} \\
a_{12} = a_{21} &= \frac{-E_p n}{a^2} - \frac{D(3-\nu)m^2 n}{2a^2} \\
a_{13} = a_{31} &= \frac{-\nu E_p m}{a} - \frac{Dm^3}{a} + \frac{G_p n^2 m}{12a}\left(\frac{h}{a}\right)^2 \\
a_{22} &= \frac{-E_p n^2}{a^2} - G_p m^2\left(1 + \left(\frac{h}{2a}\right)^2\right) \\
a_{23} = a_{32} &= \frac{-E_p nm(1+\nu)}{2a} \\
a_{33} &= -E_p m^2 - \frac{G_p n^2}{a^2}\left(1 + \frac{1}{12}\left(\frac{h}{a}\right)^2\right).
\end{aligned} \tag{6}$$

The stability matrix $B = (b_{ij})$ is (Herrmann and Armenakas, 1962):

$$\begin{aligned}
b_{11} &= -T_0 m^2 - \frac{N_0}{a^2}(1+n^2) \\
b_{12} = b_{21} &= \frac{-2N_0 n}{a^2} \\
b_{13} = b_{31} &= 0 \\
b_{22} &= -T_0 m^2 - \frac{N_0(1+n^2)}{a^2} \\
b_{23} = b_{32} &= 0 \\
b_{33} &= -T_0 m^2 - \frac{N_0 n^2}{a^2}.
\end{aligned} \tag{7}$$

The factor λ expresses hoop thrust N and axial thrust T relative to values N_0 and T_0, viz.

$$N = \lambda N_0$$
$$T = \lambda T_0. \tag{8}$$

Stiffness of the elastic continuum

Linear elastic analysis of the continuum surrounding the tube (e.g. Booker and Carter, 1984) can be used to relate the harmonic coefficients [analogous to (1, 2, 3)] of radial $\bar{W}$, circumferential $\bar{V}$ and axial displacement $\bar{U}$ at the soil–structure interface, to the normal and shear tractions $\bar{R}$, $\bar{Q}$ and $\bar{S}$, viz.

$$(\bar{R}, \bar{Q}, \bar{S})^{\mathrm{T}} = \bar{k}(\bar{W}, \bar{V}, \bar{U})^{\mathrm{T}}. \tag{9}$$

Moore (1988) describes the assessment of $\bar{k}$ where a uniform infinite elastic continuum surrounds the cylindrical cavity, and where the surrounding material is modelled as a concentric series of thick elastic tubes. (This second model is used in a later section to examine the response of stiffened tubes buried in non-uniform ground.)

Equations (9) can be adjusted according to the interface condition, to yield

$$\bar{K}(W, V, U)^{\mathrm{T}} = (\bar{R}, \bar{Q}, \bar{S})^{\mathrm{T}}. \tag{10}$$

(a) For the bonded interface condition

$$(W, V, U)^{\mathrm{T}} = (\bar{W}, \bar{V}, \bar{U})^{\mathrm{T}}$$

and therefore

$$\bar{K} = \bar{k}. \tag{11}$$

(b) For the smooth interface condition, application of

$$\bar{R} = \bar{S} = 0$$
$$W = \bar{W}$$

to (9) yields

$$\bar{K} = \begin{bmatrix} \bar{K}_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{12}$$

where

$$\bar{K}_{11} = \frac{(\bar{k}_{11}\bar{k}_{33} - \bar{k}_{23}^2)(\bar{k}_{22}\bar{k}_{33} - \bar{k}_{23}^2) - (\bar{k}_{12}\bar{k}_{33} - \bar{k}_{13}\bar{k}_{23})^2}{(\bar{k}_{22}\bar{k}_{33} - \bar{k}_{23}^2)}. \tag{13}$$

Linear eigenvalue problem

Equilibrium of tractions

$$(R, Q, S)^{\mathrm{T}} + (\bar{R}, \bar{Q}, \bar{S})^{\mathrm{T}} = (0, 0, 0)^{\mathrm{T}}$$

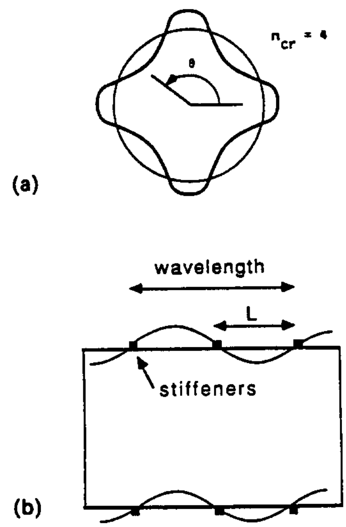


Fig. 2. Buckling deformations. (a) Circumferential deformations. (b) Axial buckling deformations.

leads to

$$(A+\lambda B+\bar{K})(W,V,U)^{\mathrm{T}} = (0,0,0)^{\mathrm{T}}. \tag{14}$$

Solution of (14) yields the linear eigenvalue problem for λ

$$\det(A+\lambda_{cr}B+\bar{K}) = 0. \tag{15}$$

Figure 2a shows the harmonic buckling response around the circumference of the structure. Each harmonic n will yield a particular load factor λ_{cr}. The lowest such factor must be determined by solving (15) for a range of n values ($n \geqslant 2$). That minimum load factor can be substituted into eqns (8) to determine the critical thrusts T_{cr} and N_{cr}.

Stability of rib-stiffened structures

Stability is influenced by the m value employed in (1), (2) and (3). For stiffened structures it is assumed that ribs prevent radial deflections and that one half-wavelength of the deformations along the pipe stretches between each pair of ribs, Fig. 2b. The harmonic m is simply a function of the rib spacing L,

$$m = \frac{\pi}{L}.$$

The solution yields the plane strain results [see the next section as well as Forrestal and Herrmann (1965) and Moore and Booker (1985)] as m approaches zero.

PARAMETRIC STUDY

Introduction

The author's primary interest is in elastically supported tubes which develop hoop thrusts from pressures acting on the external surface. These pressures may be transmitted through the elastic continuum, or they may result from a fluid acting around the structure.

The objective of this study is to determine how the use of circumferential rib stiffeners influences critical hoop thrust. A brief parametric study is undertaken to investigate the effect of stiffener spacing L, structural flexibility D, modulus and Poisson's ratio of the elastic continuum G_s and ν_s, the axial thrust T and the interface condition.

The solutions which follow have been normalized using the reference thrust adopted in previous studies (e.g. Moore and Booker, 1985)

$$N_{\text{ref}} = \frac{3D}{a^2}\left(\left(\frac{G_s a^3}{D}\right)^{2/3} + 1\right).$$

Unstiffened tubes

It is useful to first review the response of unstiffened ground supported tubes acting under plane strain conditions. Critical hoop thrust can be found by minimizing

$$\frac{Dn^2}{a^2} + \frac{2G_s a[2n(1-v_s)-(1-2v_s)]}{(n^2-1)(3-4v_s)}$$

with respect to $n \geqslant 2$ for the bonded interface condition, and

$$\frac{Dn^2}{a^2} + \frac{2G_s a n^2}{[2n(1-v_s)+(1-2v_s)](n^2-1)}$$

for the smooth interface condition (Moore and Booker, 1985). (It is assumed throughout that loads applied do not rotate as the structure deforms.) For negligible ground support, $G_s < 0.1\ D/a^3$, the circumferential harmonic leading to critical hoop thrust is $n_{cr} = 2$. For stiff ground, $G_s \geqslant 10^3\ D/a^3$, 10 or more buckle waves can form around the circumference. Increases in ground support lead to decreases in buckle wavelength. It is expected that the addition of rib stiffeners will stabilize the structure by increasing the critical harmonic number above that which occurs for the plane strain case. Typical buried structures where stability assessment is required (e.g. corrugated metal culverts and large diameter buried plastic pipes) have

$$\frac{D}{G_s a^3} \leqslant 10^{-2}.$$

Influence of axial thrust

Axial thrusts may develop in conjunction with hoop thrusts, and the effect on stability of axial thrust is investigated here. Figure 3 shows critical hoop thrust values N_{cr} for stiffener spacing $L = a/2$ evaluated for a range of normalized structural stiffnesses $D/G_s a^3$, and four values of axial thrust $T = 0$, $N/4$, $N/2$, N.

Axial thrust does reduce critical hoop thrust. For a tube loaded under plane strain conditions $T = vN$. A value of $T = N/2$ is used in the remainder of this study as this is considered to be a conservative value.

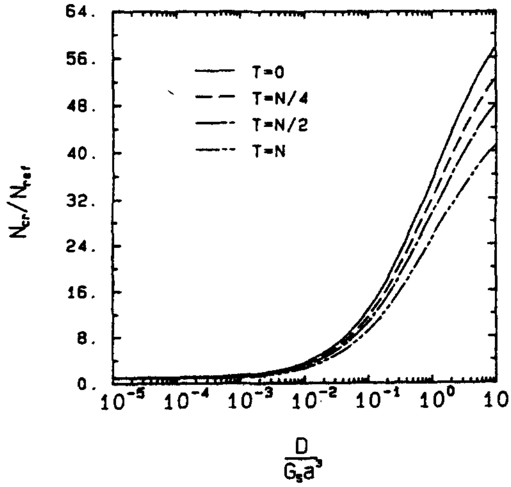


Fig. 3. Effect of axial thrust on critical hoop thrust ($v = v_s = 0.3$, $L = a/2$, bonded interface).

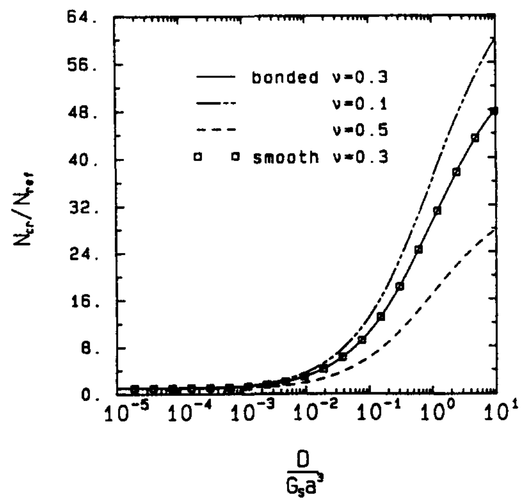


Fig. 4. Effect of Poisson's ratio and interface condition on critical hoop thrust ($L = a/2$, $T = N/2$, $\nu_s = 0.3$).

Poisson's ratio and interface condition

Figure 4 shows critical thrust N_{cr} evaluated for $L = a/2$ and $T = N/2$. Solutions are presented for bonded interface with $\nu_s = \nu = 0.3$; $\nu_s = 0.3$, $\nu = 0.1$; and $\nu_s = 0.3$, $\nu = 0.5$. The solution for smooth interface and $\nu_s = \nu = 0.3$ is also given.

The use of continuum shear modulus G_s rather than Young's modulus to normalize flexural stiffness D leads to solutions which are independent of Poisson's ratio ν_s. However, there are significant differences in solutions for different values of structural Poisson's ratio ν. An incompressible material where $\nu = 0.5$ has lowest stability.

The interface condition does not appear to be particularly important. Its significance would be greater where ground is stiff; however, the rib stiffeners are then of less significance.

Stiffener spacing

Solutions for elastically supported cylinders with stiffeners at various spacings are given in Fig. 5. Critical thrust is shown for large spacing (plane strain conditions), as well as for $L = 4a$, a, $a/2$, $a/4$ and $a/8$. A range of flexural stiffness ratios $D/G_s a^3$ are again considered.

From Fig. 5 it can be concluded that stiffeners improve the buckling capacity of structures with minimal ground support most significantly. They do little to assist where

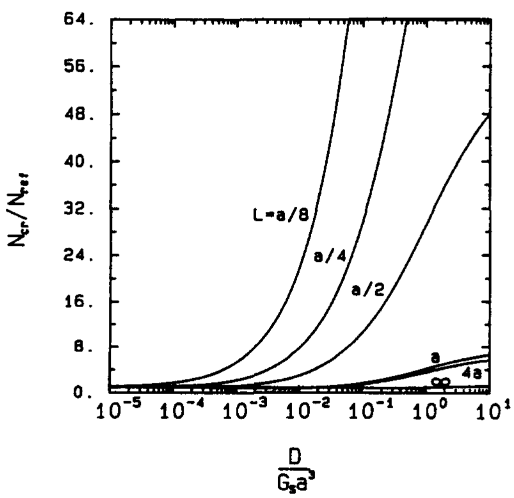


Fig. 5. Effect of stiffener spacing on critical hoop thrust ($T = N/2$, $\nu = \nu_s = 0.3$, bonded interface).

the spacing is greater than the tube radius, but can lead to substantial increases in buckling strength when spacing is less than the radius. Now typical buried flexible tubes have

$$10^{-5} < \frac{D}{G_s a^3} < 10^{-2}$$

and for these conditions the extent to which stiffeners may improve performance is variable. If the ground is very stiff ($D/G_s a^3 < 10^{-4}$) then the circumferential buckle wavelength is small and stiffeners must be very closely spaced ($L < a/8$) before buckling capacity is affected. However, for ground of lower stiffness the stiffeners modify the development of circumferential buckles when $a/8 \leqslant L \leqslant a/2$, and there may be some benefit in using rib-stiffening.

Tube buried in non-uniform ground

One situation where the elastic buckling strength of buried tubes can be inadequate occurs when there is only a limited amount of stiff backfill surrounding the structure. The structure may be supported by a thin envelope of engineered backfill which is itself surrounded by poor material of low stiffness, or the ground support above the crown may be restricted due to shallow burial.

Figure 6 shows estimates of critical thrust for a tube buried in a uniform elastic continuum, and also a tube buried in a thick cylinder of elastic material of modulus G_s (external diameter $b/a = 1.2$) which is surrounded by another material of shear modulus $G_s/10$. The load capacity of each structure has been evaluated for plane strain conditions ($L = \infty$) and for rigid ribs at spacings $L = 4a$ and $L = a/2$.

An examination of Fig. 6 shows the significant reduction in buckling strength which occurs when only a thin envelope of stiff backfill surrounds a structure with no rib supports (see also Moore *et al.*, 1988). For this particular configuration of non-uniform elastic support, the buckling strength is reduced by a factor of 3 at $D/G_s a^3 = 0.01$.

Stiffeners may be used to compensate for these strength reductions. For stiffeners spaced at $L = 4a$, the critical thrust is at least doubled in the region $D/G_s a^3 \geqslant 0.01$. However, this spacing is too great to influence buckling strength when soil stiffness is such that $D/G_s a^3 \leqslant 10^{-3}$. Use of stiffeners at the closer spacing of $L = a/2$ is sufficient to eliminate the strength reductions for the full range of $D/G_s a^3$ values.

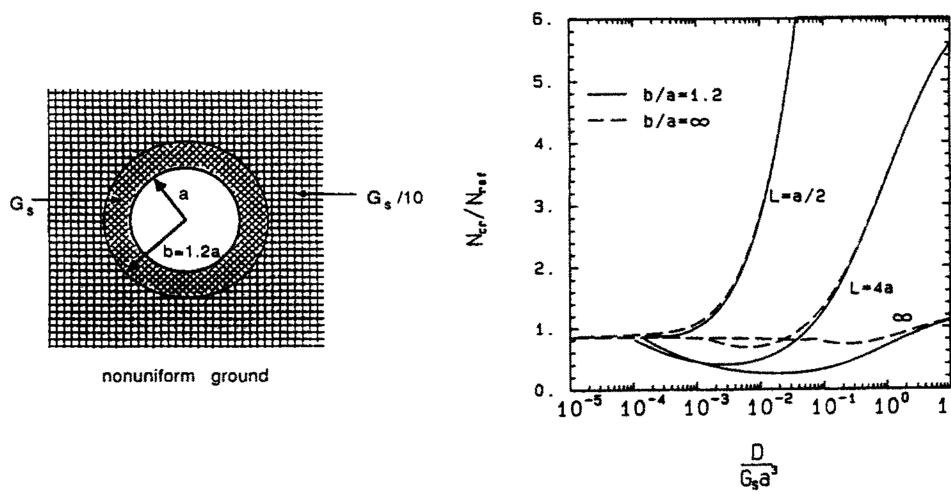


Fig. 6. Effect of rib stiffeners for a flexible tube buried in non-uniform ground ($T = N/2$, $\nu = \nu_s = 0.3$, bonded interface).

CONCLUDING DISCUSSION

The elastic buckling strength of long circular tubes, supported by an elastic continuum and stiffened by rigid circumferential ribs, has been considered. Harmonic stiffness equations for the structure and the elastic continuum supporting it, have been used to solve the linear buckling problem.

A parametric study of the problem has revealed that for deeply buried structures in ground of low stiffness ($D/G_s a^3 \geqslant 10^{-3}$) there is improvement in buckling strength if stiffener spacing is less than the tube radius. For structures buried in stiff ground, $D/G_s a^3 \leqslant 10^{-4}$, the stiffeners must be very closely spaced ($L < a/8$) before buckling strength is significantly affected.

It was demonstrated that the interface condition is not particularly important, but the axial thrust has some influence on stability. The possibility of using circumferential ribs to enhance buckling strength was investigated for circumstances where there is reduced ground support associated with reductions in quantity of stiff backfill. It appears that ribs may be successfully designed to provide the necessary strength for these situations. Further work is needed to investigate the rigid stiffener assumption, and to assess stability where the ribs have some torsional stiffness, and where the structure is a corrugated tube with anisotropic properties.

REFERENCES

Allgood, J. R. and Ciani, J. B. (1968). The influence of soil modulus on the behaviour of cylinders in sand. *Highway Research Record*, No. 249, pp. 1–13.

Booker, J. R. and Carter, J. P. (1984). Analysis of deformations caused by loadings applied to the walls of a circular tunnel. *Int. J. Num. Anal. Meth. Geomech.* **8**(5), 445–455.

Crabb, G. I. and Carder, D. R. (1985). Loading tests on buried flexible pipes to validate a new design model. *Research Report 28*, TRRL, Crowthorne.

Einstein, H. H. and Schwartz, C. W. (1979). Simplified analysis for tunnel supports. *J. Geotech. Divn, ASCE* **105**, GT4, 499–518.

Forrestal, M. J. and Herrmann, G. (1965). Buckling of a long cylindrical shell surrounded by an elastic medium. *Int. J. Solids Structures* **1**, 297–309.

Gumbel, J. E. (1983). Analysis and design of buried flexible pipes. Ph.D. Thesis, Department of Civil Engineering, University of Surrey, U.K.

Herrmann, G. and Armenakas, A. E. (1962). Dynamic behaviour of cylindrical shells under initial stress. *Proc. 4th U.S. National Congress of Applied Mechanics, ASME* **1**, 203–213.

Hoeg, K. (1968). Stresses against underground structural cylinders. *J. Soil Mech. Fndn Divn, ASCE* **94**, SM4, 833–858.

Moore, I. D. (1988). Three dimensional response of elastic tubes. *Department of Civil Engineering and Surveying Research Report No. 033.11.1988*. The University of Newcastle, ISBN 0 7259 0633 2 (see also *Int. J. Solids Structures* **26**(4), 391–400).

Moore, I. D. (1989). Elastic buckling of buried flexible tubes—A review of theory and experiment. *J. Geotech. Divn, ASCE* **115**, GT3, 340–358.

Moore, I. D. and Booker, J. R. (1985). Simplified theory for the behaviour of buried flexible cylinders under the influence of uniform hoop compression. *Int. J. Solids Structures* **21**(9), 929–941.

Moore, I. D., Haggag, A. and Selig, E. T. (1988). Buckling of cylinders supported by nonhomogeneous elastic ground. Internal Report No. TRB88-346I, Geotechnical Engineering, University of Massachussetts, U.S.A.

Selig, E. T. and Nash, W. A. (1988). Buried pipeline research needs. Final Report on N.S.F. Workshop. Internal Report NSF88-349F, Geotechnical Engineering, University of Massachussetts, U.S.A.